

AN UPPER BOUND OF THE HEAT KERNEL ALONG THE HARMONIC-RICCI FLOW

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ABSTRACT. In this paper, we first derive a Sobolev inequality along the harmonic-Ricci flow. We then prove a linear parabolic estimate based on the Sobolev inequality and Moser's iteration. As an application, we will obtain an upper bound estimate for the heat kernel under the flow.

1. INTRODUCTION

Let M^n be an n dimensional closed smooth manifold and assume $n \geq 3$. In [19], Müller studied a system of the Ricci flow coupled with a harmonic map heat flow

$$\begin{cases} \partial_t g = -2\text{Ric} + 2\alpha(t)\nabla\phi \otimes \nabla\phi, \\ \partial_t \phi = \tau_g \phi, \end{cases} \quad (1.1)$$

where $\phi(\cdot, t) : (M, g(\cdot, t)) \rightarrow (N, h)$ is a family of smooth maps between two Riemannian manifolds, both $g(\cdot, t)$ and h are Riemannian metrics, $\alpha(t)$ is a positive non-increasing function, and $\tau_g \phi$ denotes the intrinsic Laplacian of ϕ . This flow is also called harmonic-Ricci flow (cf. [2, 19, 27]). The harmonic-Ricci flow may be one of helpful tools in finding the harmonic map between two Riemannian manifolds. If the target manifold N is \mathbb{R} , the harmonic-Ricci flow reduces to the extended Ricci flow, which was first introduced by List in [17]. The extended Ricci flow is very useful in general relativity. If ϕ is a constant map, the system (1.1) degenerates to Hamilton's Ricci flow discussed widely recently, see for example the book [5] and seminal papers [4, 10, 11, 20]. Similarly as Ricci flow and the extended Ricci flow, corresponding theories for the harmonic-Ricci flow were established in [19], such as the short time existence, the \mathcal{W} entropy, the \mathcal{F} entropy, reduced length and reduced volume. Hence the harmonic-Ricci flow may be investigated through the methods used in the Ricci flow.

In this paper, along the harmonic-Ricci flow, we consider the heat kernel $G(x, t, y, s)$ which is the fundamental solution of the following heat equation

$$(\Delta - \partial_t)u(x, t) = 0. \quad (1.2)$$

The estimate for heat kernel has always been an interesting topic in the study of differential equations on manifolds. In their celebrated paper [16], Li and Yau derived some point-wise gradient estimates for the positive solutions of (1.2) on complete manifolds with fixed metric and lower bounded Ricci curvature, from which the upper and lower bounds on the heat kernel were obtained. Wang [21] proved a global gradient estimate when the

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boundary of manifold is nonconvex, and got both upper and lower bounds for the heat kernel with Neumann conditions. Later, in [1, 2, 18] the evolved metrics were studied, and some bounds on heat kernel under some geometric flows (e.g. the Ricci flow and the extended Ricci flow) were also derived with the assistance of the Sobolev inequality.

It is well-known that the Sobolev inequality is an important analytical tool in geometric analysis. Recently, there occur many interesting results on the Sobolev inequality under different geometric flows, especially the Ricci flow. In [12, 14, 22, 23, 24, 25, 26], some uniform Sobolev inequalities were proved along Ricci flow by using the monotonicity of Perelman's \mathcal{W} entropy. As a consequence, Perelman's short time non-collapsing was extended to a long time version. In particular, by the Sobolev inequality, Zhang [24] proved a global upper bound for the fundamental solution of a heat equation under backward Ricci flow with the assumption that Ricci curvature is nonnegative and the injectivity radius is bounded from below.

The main purpose of this paper is to establish the uniform Sobolev inequality and an upper bound for the heat kernel under the harmonic-Ricci flow. For convenience, we denote as in [6, 7, 17, 19] the symmetric two-tensor field S_y with components S_{ij} and its trace $S := g^{ij}S_{ij}$ by

$$S_{ij} := R_{ij} - \alpha(t)\nabla_i\phi\nabla_j\phi \quad \text{and} \quad S := R - \alpha(t)|\nabla\phi|^2,$$

where R_{ij} and R are the Ricci curvature components and the scalar curvature of (M, g) respectively. Using the monotonicity of the \mathcal{W} entropy, we obtain the following Sobolev inequality.

Theorem 1.1. *Let $(g(x, t), \phi(x, t))$ be a solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T)$ with initial value (g_0, ϕ_0) . Let A_0 and B_0 be positive numbers such that the following L^2 Sobolev inequality holds initially, i.e. for any $v \in W^{1,2}(M, g_0)$,*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g_0) \right)^{\frac{n-2}{n}} \leq A_0 \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu(g_0) + B_0 \int_M v^2 d\mu(g_0).$$

Then for all $v \in W^{1,2}(M, g(t))$ we have

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g(t)) \right)^{\frac{n-2}{n}} \leq A(t) \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu(g(t)) + B(t) \int_M v^2 d\mu(g(t)),$$

where $A(t) = Ce^{\frac{8tB_0}{nA_0}}A_0$, $B(t) = Ce^{\frac{8tB_0}{nA_0}}B_0$, and C is a positive constant depending only on A_0 , B_0 , g_0 , ϕ_0 and n . In particular, if $S \geq 0$ at the initial time, then C is a positive constant depending only on n .

Based on the Sobolev inequality and Moser's iteration, Ye [22] proved a linear parabolic estimate under the Ricci flow, which was applied to get the upper bound of curvature tensor. Jiang [13] gave a linear parabolic estimate along the Kähler-Ricci flow, from which he obtained upper bound estimates of the scalar curvature and the gradient of Ricci potential. Here from the above Sobolev inequality, we can get the following linear parabolic estimate.

Theorem 1.2. Assume that $(g(x, t), \phi(x, t))$ is a smooth solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T]$ with initial value (g_0, ϕ_0) and $S \geq 0$ at the initial time. Let f be a nonnegative Lipschitz continuous function on $M \times [0, T]$ satisfying

$$\partial_t f \leq \Delta f + af \quad (1.3)$$

on $M \times [0, T]$ in the weak sense, where $a \geq 0$. Then we have for any $0 < t \leq T$ and $p > 0$

$$\sup_{x \in M} |f(x, t)| \leq \left(C_1 a + \frac{C_2}{t} \right)^{\frac{n+2}{2p}} \left(\int_0^T \int_M f^p d\mu dt \right)^{\frac{1}{p}},$$

where C_1 and C_2 are both positive constants depending on dimension n , p , g_0 , ϕ_0 and the first eigenvalue λ_0 of \mathcal{F} entropy with respect to g_0 .

Obviously, the heat equation (1.2) is a simple linear parabolic equation and the heat kernel satisfies naturally the conditions of the above theorem. As a consequence of Theorem 1.2, it is not difficult to get an upper bound of the heat kernel under the harmonic-Ricci flow, which is similar to the upper bound in Bailesteanu [1], Bailesteanu and Tran [2], and Wang [21]. But our upper bound depends on the first eigenvalue of \mathcal{F} entropy, which is different from their results. More precisely, we prove

Theorem 1.3. Assume that $(g(x, t), \phi(x, t))$ is a smooth solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T]$ with initial value (g_0, ϕ_0) and $S \geq 0$ at the initial time. Let $G(x, t; y, s)$ be the heat kernel. Then there exists a positive constant C , which depends on dimension n , g_0 , ϕ_0 and the first eigenvalue λ_0 of \mathcal{F} entropy with respect to g_0 , such that

$$G(x, t; y, s) \leq \frac{C}{(t - s)^{\frac{n}{2}}},$$

for $\forall 0 \leq s < t \leq T$, and $\forall x, y \in M$.

Remark 1.1. Here the nonnegativity of S in our theorem is a little weaker than the positivity in Bailesteanu and Tran [2]. Indeed, the assumption can be taken away if we impose other reliance of constant C on the upper bound of time (see Corollary 3.3).

The rest of the paper is organized as follows. In section 2 we consider the \mathcal{W} entropy under the harmonic-Ricci flow and derive the Sobolev inequality by using the monotonicity of \mathcal{W} entropy. In section 3 we show the linear parabolic estimate for (1.3) and the upper bound estimate of the heat kernel along the harmonic-Ricci flow.

2. SOBOLEV INEQUALITIES UNDER THE HARMONIC-RICCI FLOW

In this section, we show a uniform log-Sobolev inequality along the harmonic-Ricci flow from the monotonicity of \mathcal{W} entropy, and then verify the equivalence of our Sobolev inequality and the uniform log-Sobolev inequality, from which we prove Theorem 1.1. As a corollary, we obtain the other uniform Sobolev inequality depending on the first eigenvalue of \mathcal{F} entropy, which will be used in the next section.

First let us introduce the definition of \mathcal{W} entropy via corresponding conjugate heat equation just as Perelman [20] has done in Ricci flow. Let $u(x, t)$ be a positive solution to the following conjugate heat equation

$$\Delta u - Su + \partial_t u = 0. \quad (2.1)$$

From the conjugate heat equation (2.1) and the harmonic-Ricci flow (1.1), we can get easily

$$\frac{d}{dt} \int_M u(x, t) d\mu(g(t)) = \int_M (\partial_t - S)u d\mu(g(t)) = - \int_M \Delta u d\mu(g(t)) = 0,$$

where we used the fact that M is closed. Therefore, without loss of generality we assume that $u(x, t)$ satisfies

$$\int_M u(x, t) d\mu(g(t)) = 1$$

for any $t \in [0, T]$. The \mathcal{W} entropy is given by a functional of the positive solution u of (2.1) as follows.

Definition 2.1. The \mathcal{W} entropy is defined as the following functional

$$\mathcal{W}(g, f, \tau) = \int_M (\tau(S + |\nabla f|^2) + f - n)u d\mu(g(t)),$$

where $f = -\ln u - \frac{n}{2}(\ln 4\pi\tau)$ and τ is a scaling factor satisfied $\frac{d\tau}{dt} = -1$.

Remark 2.1. The same definition can also be found in [17, 19]. From the relationship between f and u , \mathcal{W} entropy can also be rewritten by the function u directly as follow.

$$\mathcal{W}(g, u, \tau) := \int_M \left[\tau \left(Su + \frac{|\nabla u|^2}{u} \right) - u \ln u - \frac{n}{2} \ln(4\pi\tau)u - nu \right] d\mu(g(t)). \quad (2.2)$$

Now let us recall the following monotonicity formula, which had been proved in Theorem 5.2 of [9] for general geometric flow and Proposition 7.1 of [19](or Theorem 6.1 of [17]) for the case of constant α . We omit the details here.

Proposition 2.1. *Let $(g(x, t), \phi(x, t))$ be a solution of the harmonic-Ricci flow (1.1) and $u(x, t)$ be a positive solution of (2.1). Then \mathcal{W} entropy is non-decreasing in t . More precisely,*

$$\frac{d}{dt} \mathcal{W} = \int_M \left(2\tau |S_y + Hess(f) - \frac{g}{2\tau}|^2 + 2\tau\alpha |\tau_g \phi - \langle \nabla \phi, \nabla f \rangle|^2 - \tau \dot{\alpha} |\nabla \phi|^2 \right) u d\mu(g(t)) \geq 0.$$

To prove Theorem 1.1, we first need to prove the corresponding log-Sobolev inequality for any $t \in [0, T]$. Here we use the same method as Zhang [25] and Liu-Wang [18]. Using the monotonicity of \mathcal{W} entropy, we have the following log-Sobolev inequality.

Lemma 2.2 (Log-Sobolev Inequality). *Under the same assumptions of Theorem 1.1. Then for any $t \in [0, T]$, $v \in W^{1,2}(M, g(t))$ with $\int_M v^2 d\mu(g(t)) = 1$ and any $\epsilon > 0$, we have*

$$\int_M v^2 \ln v^2 d\mu(g(t)) \leq \epsilon^2 \int_M (4|\nabla v|^2 + Sv^2) d\mu(g(t)) - n \ln(2\epsilon) + 4(t + \epsilon^2) \frac{B_0}{A_0} + \frac{n}{2} \ln \frac{nA_0}{2e}.$$

Proof. For any fixed $t_0 \in [0, T)$ and any $\epsilon > 0$, we set $\tau(t) = \epsilon^2 + t_0 - t$. From the monotonicity of the \mathcal{W} entropy in Proposition 2.1, we get

$$\begin{aligned} \inf_{\int_M u_0 d\mu(g_0)=1} \mathcal{W}(g_0, f_0, t_0 + \epsilon^2) &\leq \mathcal{W}(g_0, \tilde{f}(\cdot, 0), t_0 + \epsilon^2) \\ &\leq \mathcal{W}(g(t_0), \tilde{f}(\cdot, t_0), \epsilon^2) \\ &= \inf_{\int_M u d\mu(g(t_0))=1} \mathcal{W}(g(t_0), f, \epsilon^2), \end{aligned} \quad (2.3)$$

where $(4\pi\tau)^{-\frac{n}{2}}e^{-\tilde{f}(\cdot, t)}$ satisfies the conjugate heat equation (2.1), f_0 and f are given by the formulas $u_0 = (4\pi(t_0 + \epsilon^2))^{-\frac{n}{2}}e^{-f_0}$ and $u = (4\pi\epsilon^2)^{-\frac{n}{2}}e^{-f}$. The last equality holds because the infimum of \mathcal{W} entropy is achieved by a minimizer $\tilde{f}(\cdot, t_0)$ (cf. Corollary 1.5.9 in [4] or section 3 in [20]). Using (2.2) we rewrite (2.3) as

$$\begin{aligned} \inf_{\int u_0 d\mu(g_0)=1} \int_M \left[(\epsilon^2 + t_0)(S + |\nabla \ln u_0|^2) - \ln u_0 - \frac{n}{2} \ln(4\pi(t_0 + \epsilon^2)) \right] u_0 d\mu(g_0) \\ \leq \inf_{\int u d\mu(g(t_0))=1} \int_M \left[\epsilon^2 (S + |\nabla \ln u|^2) - \ln u - \frac{n}{2} \ln(4\pi\epsilon^2) \right] u d\mu(g(t_0)). \end{aligned}$$

Let $v = \sqrt{u}$ and $v_0 = \sqrt{u_0}$, the above inequality leads to

$$\begin{aligned} \inf_{\int v_0^2 d\mu(g_0)=1} \int_M \left[(\epsilon^2 + t_0)(Sv_0^2 + 4|\nabla v_0|^2) - v_0^2 \ln v_0^2 \right] d\mu(g_0) - \frac{n}{2} \ln(t_0 + \epsilon^2) \\ \leq \inf_{\int v^2 d\mu(g(t_0))=1} \int_M \left[\epsilon^2 (Sv^2 + 4|\nabla v|^2) - v^2 \ln v^2 \right] d\mu(g(t_0)) - \frac{n}{2} \ln \epsilon^2. \end{aligned} \quad (2.4)$$

Notice that $\ln x$ is a concave function and $\int v_0^2 d\mu(g_0) = 1$, thus applying Jensen's inequality we deduce

$$\int_M v_0^2 \ln v_0^{q-2} d\mu(g_0) \leq \ln \int v_0^{q-2} v_0^2 d\mu(g_0),$$

where $q = \frac{2n}{n-2}$. This means

$$\int_M v_0^2 \ln v_0^2 d\mu(g_0) \leq \frac{n}{2} \ln \|v_0\|_q^2,$$

By the assumption that the Sobolev inequality holds for the initial time $t = 0$, combining with the above inequality we have

$$\int_M v_0^2 \ln v_0^2 d\mu(g_0) \leq \frac{n}{2} \ln \left(A_0 \int_M \left(|\nabla v_0|^2 + \frac{1}{4} S v_0^2 \right) d\mu(g_0) + B_0 \right).$$

Moreover, the inequality $\ln z \leq yz - \ln y - 1$ holds for any $y, z > 0$. Using it in the RHS of the above we arrive at

$$\int_M v_0^2 \ln v_0^2 d\mu(g_0) \leq \frac{n}{2} y \left(A_0 \int_M \left(|\nabla v_0|^2 + \frac{1}{4} S v_0^2 \right) d\mu(g_0) + B_0 \right) - \frac{n}{2} \ln y - \frac{n}{2}.$$

Now we choose $y = \frac{8(t_0 + \epsilon^2)}{nA_0}$, then the above inequality implies

$$\begin{aligned} \int_M v_0^2 \ln v_0^2 d\mu(g_0) &\leq (t_0 + \epsilon^2) \int_M (4|\nabla v_0|^2 + Sv_0^2) d\mu(g_0) + \frac{4(t_0 + \epsilon^2)B_0}{A_0} \\ &\quad - \frac{n}{2} \ln \frac{8(t_0 + \epsilon^2)}{nA_0} - \frac{n}{2}. \end{aligned} \quad (2.5)$$

Substituting (2.5) into (2.4), we conclude that

$$\begin{aligned} \int_M v^2 \ln v^2 d\mu(g(t_0)) &\leq \epsilon^2 \int_M (4|\nabla v|^2 + Sv^2) d\mu(g(t_0)) - n \ln(2\epsilon) \\ &\quad + \frac{4(t_0 + \epsilon^2)B_0}{A_0} + \frac{n}{2} \ln \frac{nA_0}{2e}. \end{aligned}$$

The time t_0 is arbitrary, thus the proof of the lemma is completed now. \square

In general, the log-Sobolev inequality and the Sobolev inequality are equivalent, which can be proved via the upper bound of heat kernel. More details can be found in Zhang's Theorem 4.2.1 of [26]. But the Sobolev inequality along a geometric flow is different from the general Sobolev inequality in closed manifolds. So it is necessary to provide the equivalence between our log-Sobolev inequality and Sobolev inequality here. We can give a proof of the following equivalence lemma by the trick in [3].

Lemma 2.3. *Let (M^n, g) be a closed Riemannian manifold ($n \geq 3$). Then the following inequalities are equivalent up to constants.*

(I) *Sobolev inequality: there exist positive constants A and B such that, for all $v \in W^{1,2}(M)$,*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu + B \int_M v^2 d\mu;$$

(II) *Log-Sobolev inequality: for all $v \in W^{1,2}(M)$ such that $\|v\|_2 = 1$ and all $\epsilon > 0$,*

$$\int_M v^2 \ln v^2 d\mu \leq \epsilon^2 \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu - \frac{n}{2} \ln \epsilon^2 + BA^{-1}\epsilon^2 + \frac{n}{2} \ln \frac{nA}{2e}.$$

Proof. $I \Rightarrow II$: The proof is a standard application of the Jensen's inequality. The derivation is almost same with (2.5). We only need to take $y = \frac{2\epsilon^2}{nA}$ instead, the log-Sobolev inequality will be obtained as desired.

$II \Rightarrow I$: Notice that the LHS of log-Sobolev inequality is bounded from below for all $v \in W^{1,2}(M)$. Hence, the log-Sobolev inequality implies directly that for all $v \in W^{1,2}(M)$

$$A \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu + B > 0.$$

Since the log-Sobolev inequality holds for all $\epsilon > 0$, the RHS of log-Sobolev inequality can be seen as a function of ϵ and reaches its minimum. Thus we have

$$\int_M v^2 \ln v^2 d\mu \leq \frac{n}{2} \ln \left[A \int_M \left(|\nabla v|^2 + \frac{1}{4}Sv^2 \right) d\mu + B \right], \quad (2.6)$$

for all $v \in W^{1,2}(M)$ such that $\|v\|_2 = 1$. Now we consider any function $f \in W^{1,2}(M)$. By the Kato's inequality $|\nabla|f|| \leq |\nabla f|$, we only need to prove the Sobolev inequality for all nonnegative functions. So we assume that $f \geq 0$. For the sake of conveniences, we denote

$$W(f) = \left[A \int_M \left(|\nabla f|^2 + \frac{1}{4} S f^2 \right) d\mu + B \int_M f^2 d\mu \right]^{\frac{1}{2}}.$$

Taking $v = \frac{f}{\|f\|_2}$ in (2.6) yields

$$\int_M f^2 \ln \left(\frac{f}{\|f\|_2} \right)^2 d\mu \leq n \|f\|_2^2 \ln \left(\frac{W(f)}{\|f\|_2} \right).$$

It is just (LS_2^q) in Page 1067 of [3], where $q = \frac{2n}{n-2}$. Using their method to treat the above estimate, we arrive at

$$\|f\|_2 \leq W(f)^\theta \|f\|_s^{1-\theta}, \quad (2.7)$$

where $0 < s < 2$ and $\frac{1}{2} = \frac{\theta}{q} + \frac{1-\theta}{s}$. Next we need to define a family of functions f_k for $k \in \mathbb{Z}$ by

$$f_k = \min\{(f - 2^k)^+, 2^k\},$$

where $(f - 2^k)^+ = \max\{f - 2^k, 0\}$. From the definition of f_k it is obvious to have the following estimate for any $p > 0$

$$2^{pk} \mu\{f \geq 2^{k+1}\} \leq \int_M f_k^p d\mu \leq 2^{pk} \mu\{f \geq 2^k\}. \quad (2.8)$$

Set $a_k = 2^{qk} \mu\{f \geq 2^k\}$. Combining (2.7) with (2.8), we derive

$$\begin{aligned} a_{k+1} &\leq 2^{q(k+1)-2k} \int_M f_k^2 d\mu \\ &\leq 2^{q(k+1)-2k} W(f_k)^{2\theta} \|f_k\|_s^{2(1-\theta)} \\ &\leq 2^q W(f_k)^{2\theta} a_k^{\frac{2(1-\theta)}{s}}. \end{aligned}$$

Consequently, by the Hölder inequality we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} a_k &= \sum_{k \in \mathbb{Z}} a_{k+1} \\ &\leq \sum_{k \in \mathbb{Z}} 2^q W(f_k)^{2\theta} a_k^{\frac{2(1-\theta)}{s}} \\ &\leq 2^q \left(\sum_{k \in \mathbb{Z}} W(f_k)^2 \right)^\theta \left(\sum_{k \in \mathbb{Z}} a_k^{\frac{2}{s}} \right)^{1-\theta} \\ &\leq 2^q \left(\sum_{k \in \mathbb{Z}} W(f_k)^2 \right)^\theta \left(\sum_{k \in \mathbb{Z}} a_k \right)^{\frac{2(1-\theta)}{s}}, \end{aligned}$$

where the last inequality follows from $0 < s < 2$. This leads to

$$\sum_{k \in \mathbb{Z}} a_k \leq 2^{\frac{q(q-s)}{2-s}} \left(\sum_{k \in \mathbb{Z}} W(f_k)^2 \right)^{\frac{q}{2}}. \quad (2.9)$$

Moreover, it follows from the definition of a_k that

$$\begin{aligned}
\int_M f^q d\mu &= \sum_{k \in \mathbb{Z}} \int_{2^k \leq f \leq 2^{k+1}} f^q d\mu \\
&\leq \sum_{k \in \mathbb{Z}} 2^{q(k+1)} \left(\mu(f \geq 2^k) - \mu(f \geq 2^{k+1}) \right) \\
&= (2^q - 1) \sum_{k \in \mathbb{Z}} a_k.
\end{aligned} \tag{2.10}$$

Now the rest of proof is only need to control the term $W(f_k)$ in (2.9). We have the following key estimate.

CLAIM. If $\frac{A}{4}S + B \geq 0$, then for any $0 \leq f \in W^{1,2}(M)$ we have

$$\sum_{k \in \mathbb{Z}} W(f_k)^2 \leq W(f)^2.$$

Firstly, we note that

$$\begin{aligned}
\int_M |f_k|^2 d\mu &= 2 \int_0^{2^{k+1}-2^k} t^{p-1} \mu(f - 2^k \geq t) dt \\
&= 2 \int_{2^k}^{2^{k+1}} (s - 2^k)^{p-1} \mu(f \geq s) ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \int_M |f_k|^2 d\mu &= 2 \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} (s - 2^k) \mu(f \geq s) ds \\
&= \sup_{k \in \mathbb{Z}} \left\{ \sup_{s \in [2^k, 2^{k+1}]} \frac{s - 2^k}{s} \right\} \left(2 \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} s \mu(f \geq s) ds \right) \\
&\leq \frac{1}{2} \int_M f^2 d\mu.
\end{aligned}$$

Because of the assumption that $\frac{A}{4}S + B \geq 0$, we can denote $(\frac{A}{4}S + B)d\mu$ as a new measure. By the same method as above we can derive

$$\sum_{k \in \mathbb{Z}} \int_M \left(\frac{A}{4}S + B \right) f_k^2 d\mu \leq \frac{1}{2} \int_M \left(\frac{A}{4}S + B \right) f^2 d\mu. \tag{2.11}$$

In addition, it also holds that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \int_M |\nabla f_k|^2 d\mu &= \sum_{k \in \mathbb{Z}} \int_{2^k \leq f \leq 2^{k+1}} |\nabla f|^2 d\mu \\
&= \int_M |\nabla f|^2 d\mu.
\end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12) yields

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} W(f_k)^2 &= \sum_{k \in \mathbb{Z}} \left(A \int_M \left(|\nabla f_k|^2 + \frac{1}{4} S f_k^2 \right) d\mu + B \int_M f_k^2 d\mu \right) \\
&\leq A \sum_{k \in \mathbb{Z}} \int_M |\nabla f_k|^2 d\mu + \sum_{k \in \mathbb{Z}} \int_M \left(\frac{A}{4} S + B \right) f_k^2 d\mu \\
&\leq A \int_M |\nabla f|^2 d\mu + \frac{1}{2} \int_M \left(\frac{A}{4} S + B \right) f^2 d\mu \\
&\leq W(f)^2.
\end{aligned}$$

Hence the proof of the claim has been completed. Now we can use the claim to finish the proof of lemma. One should be careful because the assumption in the claim does not have to be true. But it has no effect on the final proof. Since M is a closed manifold, there exists a nonnegative constant S_0 such that $S + S_0 \geq 0$. By combining (2.9) and (2.10), we deduce

$$\begin{aligned}
\int_M f^q d\mu &\leq (2^q - 1) 2^{\frac{q(q-s)}{2-s}} \left(\sum_{k \in \mathbb{Z}} W(f_k)^2 \right)^{\frac{q}{2}} \\
&\leq (2^q - 1) 2^{\frac{q(q-s)}{2-s}} \left(\sum_{k \in \mathbb{Z}} \left(A \int_M \left(|\nabla f_k|^2 + \frac{S + S_0}{4} f_k^2 \right) d\mu + B \int_M f_k^2 d\mu \right) \right)^{\frac{q}{2}} \\
&\leq (2^q - 1) 2^{\frac{q(q-s)}{2-s}} \left(W(f)^2 + A \int_M \frac{S_0}{4} f^2 d\mu \right)^{\frac{q}{2}}, \tag{2.13}
\end{aligned}$$

where the last inequality holds due to the claim. Note that the LHS of (2.6) is bounded from below for all $v \in W^{1,2}(M)$, it implies that

$$\lambda_1 = \inf \left\{ W(f)^2 \mid \int_M f^2 d\mu = 1, f \in W^{1,2}(M) \right\} > 0,$$

i.e.

$$\int_M f^2 d\mu \leq \lambda_1^{-1} W(f)^2, \tag{2.14}$$

for any $f \in W^{1,2}(M)$. Finally, the Sobolev inequality follows from (2.13) and (2.14). We complete the proof of the lemma. \square

Therefore, Theorem 1.1 follows directly from Lemma 2.2 and Lemma 2.3.

Proof of Theorem 1.1. Note that our log-Sobolev inequality in Lemma 2.2 just has one more term $4tB_0A_0^{-1}$ than Lemma 2.3. Applying the same arguments with the second part in the proof of Lemma 2.3 to our log-Sobolev inequality, we have

$$\int_M f^q d\mu(g(t)) \leq (2^q - 1) 2^{\frac{q(q-s)}{2-s}} e^{\frac{8tB_0}{A_0(n-2)}} \left(1 + \frac{A_0 S_0}{4\lambda_1(t)} \right)^{\frac{q}{2}} W(f)^q,$$

for any $f \in W^{1,2}(M, g(t))$. Here

$$W(f) = \left[A_0 \int_M \left(|\nabla f|^2 + \frac{1}{4} S f^2 \right) d\mu(g(t)) + B_0 \int_M f^2 d\mu(g(t)) \right]^{\frac{1}{2}},$$

$$\lambda_1(t) = \inf \left\{ W(f)^2 \mid \int_M f^2 d\mu(g(t)) = 1, f \in W^{1,2}(M, g(t)) \right\},$$

and other symbols are all same as above.

By the definitions of \mathcal{F} entropy and $W(f)$, we can see that $\lambda_1(t)$ is linearly related to the first eigenvalue of \mathcal{F} entropy. On the other hand, the first eigenvalue of \mathcal{F} entropy is monotone non-decreasing in time (cf. Proposition 3.3 [19]). So $\lambda_1(t)$ is also non-decreasing in time. It follows that our uniform Sobolev inequality holds, and we get

$$A(t) = (2^q - 1)^{\frac{2}{q}} 2^{\frac{2(q-s)}{2-s}} e^{\frac{8tB_0}{nA_0}} \left(1 + \frac{A_0 S_0}{4\lambda_1(0)} \right) A_0,$$

and

$$B(t) = (2^q - 1)^{\frac{2}{q}} 2^{\frac{2(q-s)}{2-s}} e^{\frac{8tB_0}{nA_0}} \left(1 + \frac{A_0 S_0}{4\lambda_1(0)} \right) B_0,$$

where S_0 depends on g_0 and ϕ_0 , and $\lambda_1(0)$ depends on A_0, B_0, g_0 and ϕ_0 . Thus the proof of Theorem 1.1 is completed now. \square

From Theorem 1.1 we can obtain a uniform Sobolev inequality along the harmonic-Ricci flow under the assumption of positive first eigenvalue of \mathcal{F} entropy. Recall that λ_0 is the first eigenvalue of \mathcal{F} entropy with respect to the initial metric g_0 , i.e.

$$\lambda_0 = \inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + S v^2) d\mu(g_0).$$

The \mathcal{F} entropy corresponds to Perelman's \mathcal{F} entropy for the Ricci flow introduced in [20]. Similarly as the Ricci flow, the harmonic-Ricci flow can be interpreted as the gradient flow of \mathcal{F} entropy modulo a pull-back by a family of diffeomorphisms. The eigenvalue of \mathcal{F} entropy is a very powerful tool for the research on Ricci flow and Riemannian manifolds. More results can be found in [8, 15].

Since (M, g_0) is a closed Riemannian manifold of dimension $n \geq 3$, the Sobolev inequality holds, i.e. there exist positive constants A and B depending only on the initial metric g_0 such that, for any $v \in W^{1,2}(M)$,

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g_0) \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla v|^2 d\mu(g_0) + B \int_M v^2 d\mu(g_0). \quad (2.15)$$

If $\lambda_0 > 0$, combining with Sobolev inequality (2.15), we have

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g_0) \right)^{\frac{n-2}{n}} \leq \tilde{A}_0 \int_M \left(|\nabla v|^2 + \frac{1}{4} S v^2 \right) d\mu(g_0)$$

where \tilde{A}_0 depends only on initial metric g_0, ϕ_0 and λ_0 . This means that the assumption of Sobolev inequality in Theorem 1.1 at initial time holds with $B_0 = 0$. Hence, Theorem 1.1 gives us the following result.

Corollary 2.4. *Let $(g(x, t), \phi(x, t))$ be a solution of the harmonic-Ricci flow (1.1) in $M^n \times [0, T)$ with initial value (g_0, ϕ_0) . Assume that the first eigenvalue λ_0 of \mathcal{F} entropy with respect to the initial metric g_0 is positive. Then there exists a positive constant A , depending only on n , g_0 , ϕ_0 and λ_0 , such that for all $v \in W^{1,2}(M, g(t))$, $t \in [0, T)$, it holds that*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu(g(t)) \right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla v|^2 + \frac{1}{4} S v^2 \right) d\mu(g(t)). \quad (2.16)$$

Remark 2.2. The analogous uniform Sobolev inequalities were given in [25] for the Ricci flow and [18] for the extended Ricci flow.

3. THE PROOF OF THEOREM 1.2 AND THEOREM 1.3

In this section, we first prove the linear parabolic estimate under the harmonic-Ricci flow with the help of the above Sobolev inequality (2.16) and Moser's iteration. As a result, we derive an upper bound for the heat kernel, which is similar to the one known for the fixed metric case.

Proof of Theorem 1.2. For any constant $p \geq 1$, it follows from (1.3) that

$$\int_M f^p \partial_t f d\mu - \int_M f^p \Delta f d\mu \leq a \int_M f^{p+1} d\mu,$$

where the volume measure $d\mu = d\mu(g(t))$ for simplicity, and the same symbol will also be used in the rest of the proof. Integrating by parts, we have

$$\frac{1}{p+1} \int_M \partial_t f^{p+1} d\mu + \frac{4p}{(p+1)^2} \int_M \left| \nabla f^{\frac{p+1}{2}} \right|^2 d\mu \leq a \int_M f^{p+1} d\mu.$$

Since $\partial_t d\mu = -S d\mu$ and $4p \geq 2(p+1)$ for all $p \geq 1$, multiplying both sides by $p+1$, we get

$$\partial_t \int_M f^{p+1} d\mu + \int_M S f^{p+1} d\mu + 2 \int_M \left| \nabla f^{\frac{p+1}{2}} \right|^2 d\mu \leq a(p+1) \int_M f^{p+1} d\mu.$$

Notice that the condition $S \geq 0$, then we have

$$\partial_t \int_M f^{p+1} d\mu + \frac{1}{2} \int_M \left(S f^{p+1} + 4 \left| \nabla f^{\frac{p+1}{2}} \right|^2 \right) d\mu \leq a(p+1) \int_M f^{p+1} d\mu. \quad (3.1)$$

Next for any $0 < \tau < \sigma < T$ we define

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq \tau, \\ (t - \tau)/(\sigma - \tau) & \tau \leq t \leq \sigma, \\ 1 & \sigma \leq t \leq T. \end{cases}$$

Multiplying (3.1) by ψ , we obtain

$$\partial_t \left(\psi \int_M f^{p+1} d\mu \right) + \frac{1}{2} \psi \int_M \left(S f^{p+1} + 4 \left| \nabla f^{\frac{p+1}{2}} \right|^2 \right) d\mu \leq [a(p+1)\psi + \psi'] \int_M f^{p+1} d\mu.$$

Integrating this with respect to t yields

$$\begin{aligned} \sup_{\sigma \leq t \leq T} \int_M f^{p+1} d\mu + \frac{1}{2} \int_{\sigma}^T \int_M \left(S f^{p+1} + 4 \left| \nabla f^{\frac{p+1}{2}} \right|^2 \right) d\mu dt \\ \leq 2 \left[a(p+1) + \frac{1}{\sigma - \tau} \right] \int_{\tau}^T \int_M f^{p+1} d\mu dt. \end{aligned}$$

By the assumption that $S \geq 0$ at the initial time, the first eigenvalue λ_0 of \mathcal{F} entropy with respect to the initial metric g_0 is positive. Applying Hölder inequality, the above estimate and the Sobolev inequality (2.16), we deduce

$$\begin{aligned} \int_{\sigma}^T \int_M f^{(p+1)(1+\frac{2}{n})} d\mu dt &\leq \int_{\sigma}^T \left(\int_M f^{p+1} d\mu \right)^{\frac{2}{n}} \left(\int_M f^{(p+1)\frac{n}{n-2}} d\mu \right)^{\frac{n-2}{n}} dt \\ &\leq \sup_{\sigma \leq t \leq T} \left(\int_M f^{p+1} d\mu \right)^{\frac{2}{n}} \int_{\sigma}^T A \int_M \left(S f^{p+1} + 4 \left| \nabla f^{\frac{p+1}{2}} \right|^2 \right) d\mu dt \\ &\leq 4^{1+\frac{1}{n}} A \left[a(p+1) + \frac{1}{\sigma - \tau} \right]^{1+\frac{2}{n}} \left(\int_{\tau}^T \int_M f^{p+1} d\mu dt \right)^{1+\frac{2}{n}}. \end{aligned}$$

Set

$$H(p, \tau) = \left(\int_{\tau}^T \int_M f^p d\mu dt \right)^{\frac{1}{p}},$$

for any $p \geq 2$ and $0 < \tau < T$. So we get

$$H(p(1 + \frac{2}{n}), \sigma) \leq \left(4^{1+\frac{1}{n}} A \right)^{\frac{1}{p(1+\frac{2}{n})}} \left(ap + \frac{1}{\sigma - \tau} \right)^{\frac{1}{p}} H(p, \tau). \quad (3.2)$$

Now we fix $0 < t_0 < t_1 < T$, $p_0 \geq 2$. Let $\chi = 1 + \frac{2}{n}$, $p_k = p_0 \chi^k$, $\tau_k = t_0 + (1 - \frac{1}{\chi^k})(t_1 - t_0)$. Then it follows from (3.2) that

$$H(p_{k+1}, \tau_{k+1}) \leq \left(4^{1+\frac{1}{n}} A \right)^{\frac{1}{p_{k+1}}} \left(ap_k + \frac{\chi^k}{t_1 - t_0} \frac{\chi}{\chi - 1} \right)^{\frac{1}{p_k}} H(p_k, \tau_k).$$

Hence by iteration, we arrive at

$$H(p_{m+1}, \tau_{m+1}) \leq \left(4^{1+\frac{1}{n}} A \right)^{\sum_{k=0}^m \frac{1}{p_{k+1}}} \left(ap_0 + \frac{1}{t_1 - t_0} \frac{\chi}{\chi - 1} \right)^{\sum_{k=0}^m \frac{1}{p_k}} \chi^{\sum_{k=0}^m \frac{k}{p_k}} H(p_0, \tau_0).$$

Letting $m \rightarrow \infty$, we obtain

$$H(p_{\infty}, \tau_{\infty}) \leq C_0 \left[ap_0 + \frac{n+2}{2(t_1 - t_0)} \right]^{\frac{n+2}{2p_0}} H(p_0, \tau_0).$$

for all $p_0 \geq 2$. This means

$$\sup_{(x,t) \in M \times [t_1, T]} |f(x, t)| \leq \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{2p_0}} \left(\int_{t_0}^T \int_M f^{p_0} d\mu dt \right)^{\frac{1}{p_0}}, \quad (3.3)$$

where C_1 and C_2 are both positive constants depending on dimension n , p_0 , initial metric g_0 , ϕ_0 and λ_0 . Moreover, for $0 < p < 2$, we set

$$h(s) = \sup_{(x,t) \in M \times [s,T]} |f(x,t)|.$$

Combining Young inequality with (3.3), we deduce

$$\begin{aligned} h(t_1) &\leq \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{4}} \left(\int_{t_0}^T \int_M f^2 d\mu dt \right)^{\frac{1}{2}} \\ &\leq h(t_0)^{\frac{2-p}{2}} \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{4}} \left(\int_{t_0}^T \int_M f^p d\mu dt \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} h(t_0) + \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{2p}} \left(\int_{t_0}^T \int_M f^p d\mu dt \right)^{\frac{1}{p}} \end{aligned}$$

Now we use the iteration method again. Fix $0 < t_0 < t_1 < T$, for some $0 < \theta < 1$, we let $x_k = t_1 - (1 - \theta^k)(t_1 - t_0)$. Then by iteration

$$\begin{aligned} h(t_1) = h(x_0) &\leq \frac{1}{2^k} h(x_k) + \left(\int_{t_0}^T \int_M f^p d\mu dt \right)^{\frac{1}{p}} \sum_{i=0}^{k-1} \frac{1}{2^i} \left(C_1 a + \frac{C_2}{x_i - x_{i+1}} \right)^{\frac{n+2}{2p}} \\ &\leq \frac{1}{2^k} h(x_k) + \left(\int_{t_0}^T \int_M f^p d\mu dt \right)^{\frac{1}{p}} \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{2p}} \sum_{i=0}^{k-1} \left(2\theta^{\frac{n+2}{2p}} \right)^{-i} \end{aligned}$$

Choose $0 < \theta < 1$ such that $2\theta^{\frac{n+2}{2p}} > 1$, that is, $\frac{1}{2} < \theta^{\frac{n+2}{2p}} < 1$. Taking $k \rightarrow \infty$, we have

$$h(t_1) \leq \left(C_1 a + \frac{C_2}{t_1 - t_0} \right)^{\frac{n+2}{2p}} \left(\int_{t_0}^T \int_M f^p d\mu dt \right)^{\frac{1}{p}}, \quad (3.4)$$

for all $0 < p < 2$ and $0 < t_0 < t_1 < T$. By (3.3) and (3.4) together, as $t_0 \rightarrow 0$, it follows that

$$h(t_1) \leq \left(C_1 a + \frac{C_2}{t_1} \right)^{\frac{n+2}{2p}} \left(\int_0^T \int_M f^p d\mu dt \right)^{\frac{1}{p}}, \quad \forall p > 0$$

where C_1 and C_2 are both positive constants depending on dimension n , p , initial metric g_0 , ϕ_0 and λ_0 . Thus we complete the proof now. \square

Note that the heat equation (1.2) is a linear parabolic equation. The above proof for Theorem 1.2 can be applied to the heat equation almost verbatim. Thus it is not difficult to get the following corollary, which will be used to determine the upper bound of the heat kernel later.

Corollary 3.1. *Assume that $(g(x,t), \phi(x,t))$ is a smooth solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T]$ with the initial value (g_0, ϕ_0) and $S \geq 0$ at the initial time. Let u be a nonnegative smooth solution to the heat equation (1.2) on $M \times [0, T]$. Then for any $0 \leq s < t \leq T$ and $p > 0$ we have*

$$\sup_{x \in M} |u(x,t)| \leq \frac{C}{(t-s)^{\frac{n+2}{2p}}} \left(\int_s^t \int_M u(x,\tau)^p d\mu d\tau \right)^{\frac{1}{p}},$$

where C is a positive constant depending on dimension n , p , g_0 , ϕ_0 and λ_0 .

In fact, nonnegativity of S in the conditions of Theorem 1.2 can be removed. At this time the parabolic estimate will also depend on the negative lower bound of S . By the similar arguments of Theorem 1.2 we can obtain the following estimate.

Corollary 3.2. *Assume that $(g(x, t), \phi(x, t))$ is a smooth solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T]$ with initial value (g_0, ϕ_0) , $S \geq -S_0$ for a nonnegative constant S_0 at the initial time, and the first eigenvalue λ_0 of \mathcal{F} entropy with respect to g_0 is positive. Let f be a nonnegative Lipschitz continuous function on $M \times [0, T]$ satisfying*

$$\partial_t f \leq \Delta f + af$$

on $M \times [0, T]$ in the weak sense, where $a \geq 0$. Then we have for any $0 < t \leq T$ and $p > 0$

$$\sup_{x \in M} |f(x, t)| \leq \left(C_0 S_0 + C_1 a + \frac{C_2}{t} \right)^{\frac{n+2}{2p}} \left(\int_0^T \int_M f^p d\mu dt \right)^{\frac{1}{p}},$$

where C_0 , C_1 and C_2 are all positive constants depending on dimension n , p , g_0 , ϕ_0 and λ_0 .

Now we turn to control the heat kernel, which is easy to carry out by means of using the result in Corollary 3.1.

Proof of Theorem 1.3. By the definition of heat kernel, we know the fact

$$\partial_t G(x, t; y, s) - \Delta_x G(x, t; y, s) = 0.$$

Combining with the assumption of $S \geq 0$, we have

$$\partial_t \int_M G(x, t; y, s) d\mu(x, t) = \int_M [\Delta_x G(x, t; y, s) - SG(x, t; y, s)] d\mu(x, t) \leq 0.$$

It implies that the above integral of heat kernel is non-increasing in t . So we derive

$$\int_M G(x, t; y, s) d\mu(x, t) \leq \int_M G(x, s; y, s) d\mu(x, s) = 1,$$

for $\forall 0 \leq s < t \leq T$. Therefore, by Corollary 3.1, it follows that

$$G(x, t; y, s) \leq \frac{C}{(t-s)^{\frac{n+2}{2}}} \int_s^t \int_M G(x, \tau; y, s) d\mu(x, \tau) d\tau \leq \frac{C}{(t-s)^{\frac{n}{2}}},$$

where C is a positive constant depending on dimension n , g_0 , ϕ_0 and λ_0 . \square

Thanks to Corollary 3.2, the same upper bound of heat kernel can be given without the assumption of the nonnegativity of S but at the cost of the dependence on the upper bound for time.

Corollary 3.3. *Assume that $(g(x, t), \phi(x, t))$ is a smooth solution to the harmonic-Ricci flow (1.1) in $M^n \times [0, T]$ with initial value (g_0, ϕ_0) and the first eigenvalue λ_0 of \mathcal{F} entropy*

with respect to g_0 is positive. Let $G(x, t; y, s)$ be the heat kernel. Then for $\forall x, y \in M$ and $\forall 0 \leq s < t \leq T$, we have

$$G(x, t; y, s) \leq \frac{C}{(t-s)^{\frac{n}{2}}},$$

where C is a positive constant, which depends on dimension n , g_0 , ϕ_0 , λ_0 and T .

Proof. By Theorem 1.3, we only need to prove the estimate for the case that S has negative minimum at the initial time. We can assume that $S_0 = -\min_{x \in (M, g_0)} S(x, 0) > 0$. From the definition of heat kernel and Corollary 3.2, we have

$$\sup_{x \in M} |G(x, t; y, s)| \leq \left(C_0 S_0 + \frac{C_2}{t-s} \right)^{\frac{n+2}{2}} \int_s^t \int_M G(x, \tau; y, s) d\mu(x, \tau) d\tau, \quad (3.5)$$

for $\forall 0 \leq s < t \leq T$. In addition, since $S > -S_0$ for any time $t \in [0, T]$, we deduce

$$\begin{aligned} \partial_t \int_M G(x, t; y, s) d\mu(x, t) &= \int_M [\Delta_x G(x, t; y, s) - S G(x, t; y, s)] d\mu(x, t) \\ &\leq S_0 \int_M G(x, t; y, s) d\mu(x, t). \end{aligned}$$

Integrating from s to τ gives

$$\int_M G(x, \tau; y, s) d\mu(x, \tau) \leq e^{S_0(\tau-s)} \int_M G(x, s; y, s) d\mu(x, s) = e^{S_0(\tau-s)},$$

for $\forall 0 \leq s < \tau \leq T$. Therefore, combining with (3.5), we conclude that

$$G(x, t; y, s) \leq \left(C_0 S_0 + \frac{C_2}{t-s} \right)^{\frac{n+2}{2}} \int_s^t e^{S_0(\tau-s)} d\tau \leq \frac{C}{(t-s)^{\frac{n}{2}}},$$

where the last inequality follows from $0 < t-s \leq T$ and C is a positive constant depending on dimension n , g_0 , ϕ_0 , λ_0 and T . \square

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